Kakutani's Splitting Procedure for Multiscale Substitution Schemes

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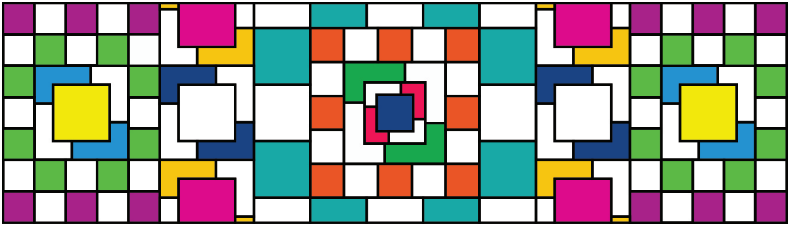
Model Sets and Aperiodic Order
Durham University, UK, September 2018



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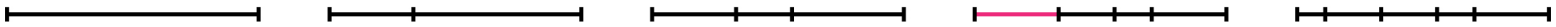

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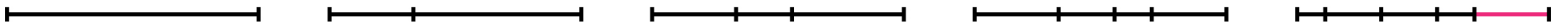

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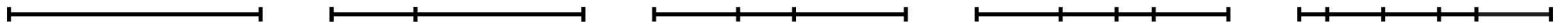

Theorem (Kakutani, 1975)

For all $\alpha \in (0, 1)$, the α-Kakutani sequence of partitions is uniformly distributed in $\mathcal{I}$.

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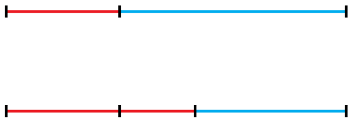

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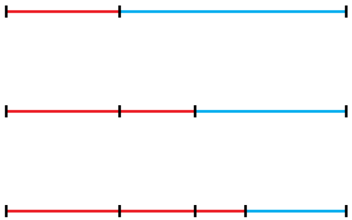

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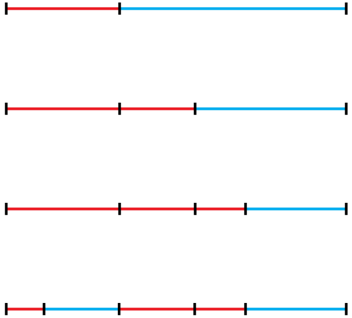

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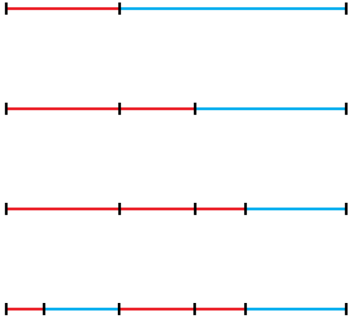

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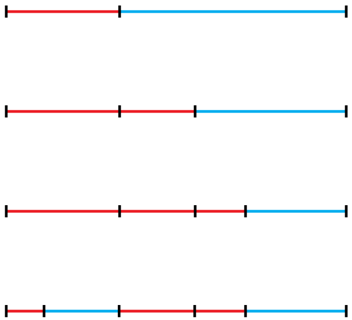

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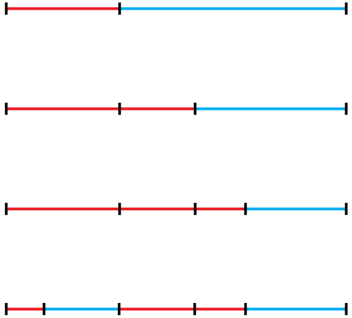

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3. In case the limits exist, are they necessarily the same?

Multiscale substitution schemes

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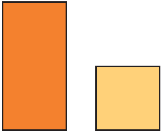

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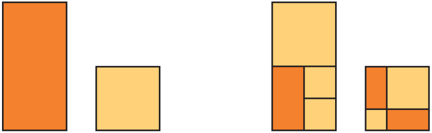

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which tile $\mathcal{T}_i$, allowing isometries.

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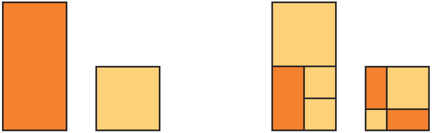

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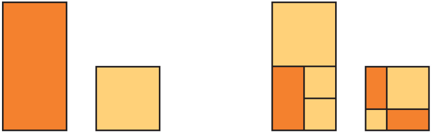

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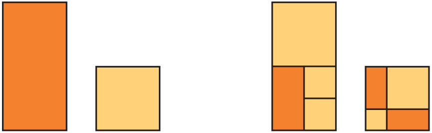

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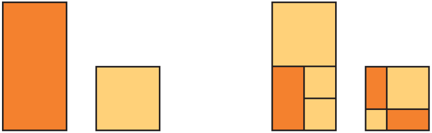

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- Tiles in $\mathcal{A}(\mathcal{T}_i)$ with label j are called **tiles of type** j.
- A scheme is **irreducible** if $\mathcal{A}(\mathcal{T}_i)$ contains tiles of type j for all i, j.

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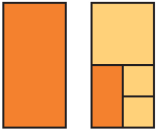

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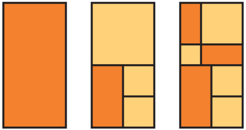

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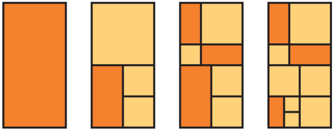

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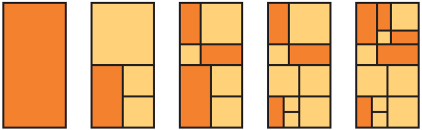

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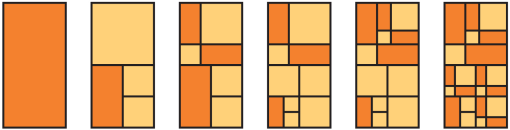

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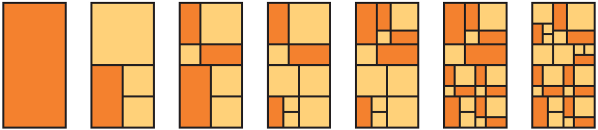

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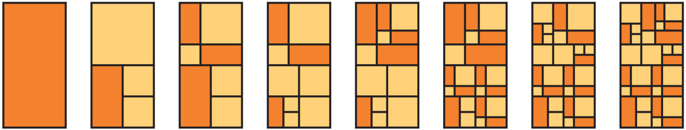

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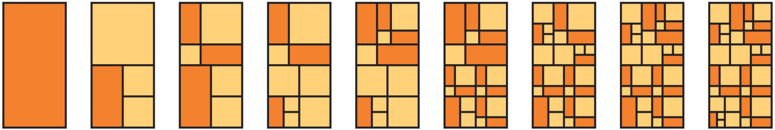

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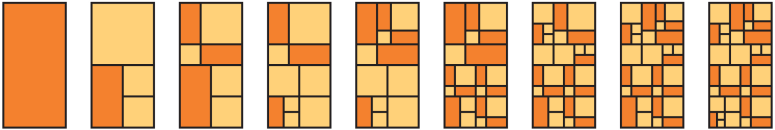

Example: The $\frac{1}{3}$-Kakutani sequence is generated by a scheme on $\mathcal{F} = \{\mathcal{I}\}$, with substitution rule $\mathcal{SR}(\mathcal{I}) = \left(\frac{1}{3}\mathcal{I}, \frac{2}{3}\mathcal{I}\right)$

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This is equivalent to the weak-* convergence of the normalized sampling measures

$$\frac{1}{|x_n|} \sum_{x \in x_n} \delta_x$$

to the normalized Lebesgue measure on U, where δ_x is the Dirac measure concentrated at x.

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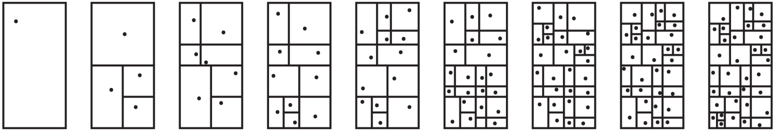

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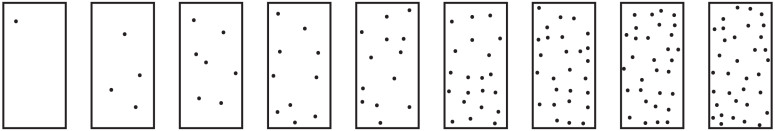

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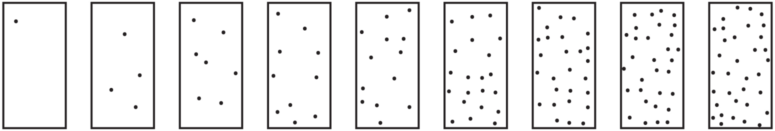

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Theorem

Let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be a set of prototiles and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $\mathcal{T}_i \in \mathcal{F}$ generated by an irreducible multiscale substitution scheme on $\mathcal{F}$. Then $\{\pi_m\}$ is uniformly distributed in $\mathcal{T}_i$.

Tile counting argument implies uniform distribution

Lemma

Let $\{\gamma_m\}$ be a sequence of partitions of $\mathcal{T}_i \in \mathcal{F}$ generated by a multiscale substitution scheme on $\mathcal{F}$, such that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ so all tiles in γ_m are of diameter less than ε for all $m \geq m_0$. Assume there exists a marking sequence $\{x_m\}$ of $\{\gamma_m\}$ such that for any tile $T \in \mathcal{A}(\mathcal{T}_i)$

$$\lim_{m\to\infty} \frac{|\{x_m \cap T\}|}{|x_m|} = \frac{\mathrm{vol}\, T}{\mathrm{vol} \mathcal{T}_i}.$$

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Tile counting argument implies uniform distribution

Lemma

Let $\{\gamma_m\}$ be a sequence of partitions of $\mathcal{T}_i \in \mathcal{F}$ generated by a multiscale substitution scheme on $\mathcal{F}$, such that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ so all tiles in γ_m are of diameter less than ε for all $m \geq m_0$. Assume there exists a marking sequence $\{x_m\}$ of $\{\gamma_m\}$ such that for any tile $T \in \mathcal{A}(\mathcal{T}_i)$

$$\lim_{m\to\infty} \frac{|\{x_m \cap T\}|}{|x_m|} = \frac{\mathrm{vol}\, T}{\mathrm{vol} \mathcal{T}_i}.$$

Then $\{\gamma_m\}$ is uniformly distributed in $\mathcal{T}_i$.

Counting of tiles is done using **directed weighted metric graphs**.

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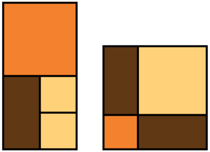
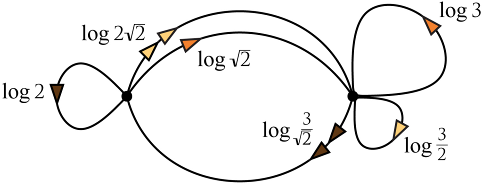
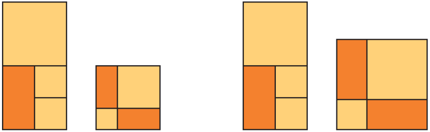




Equivalent and normalized schemes

Two schemes on $\mathcal{F}^1 = (\mathcal{T}_1, \dots, \mathcal{T}_n)$ and $\mathcal{F}^2 = (\lambda_1 \mathcal{T}_1, \dots, \lambda_n \mathcal{T}_n)$ are **equivalent** if the substitution rules are the same up to rescaling.



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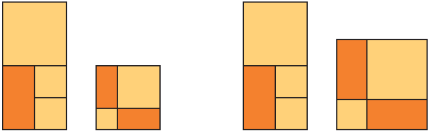

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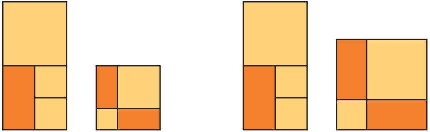

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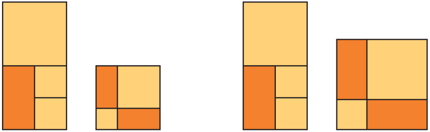

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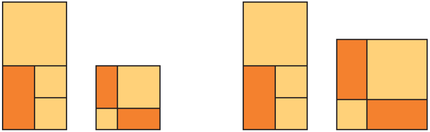

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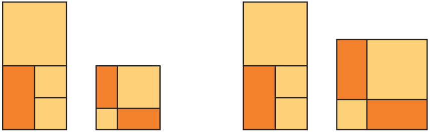

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The β_{ij}'s are called the **constants of substitution**.

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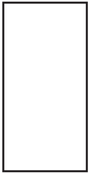

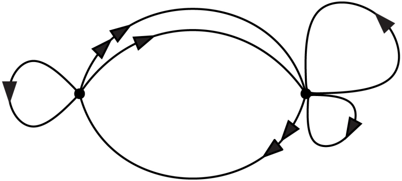

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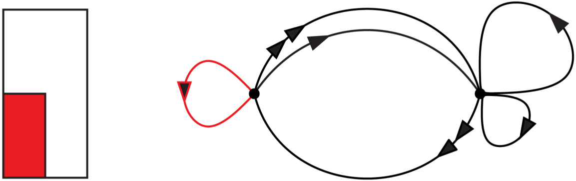

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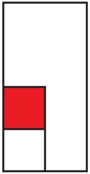

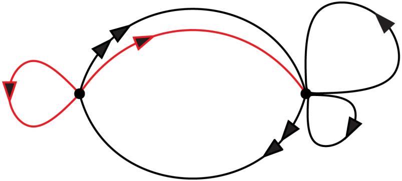

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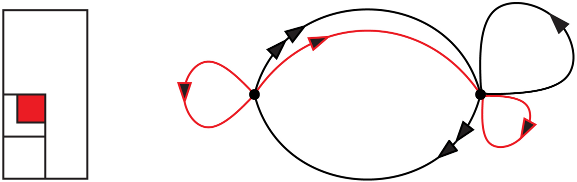

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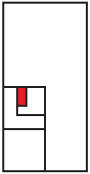

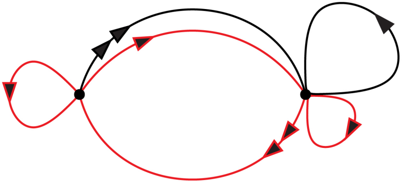

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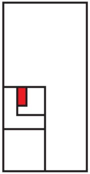

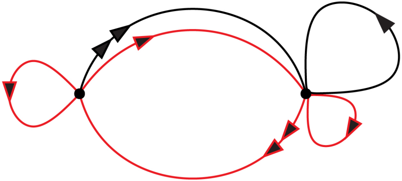

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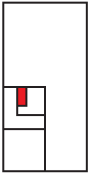

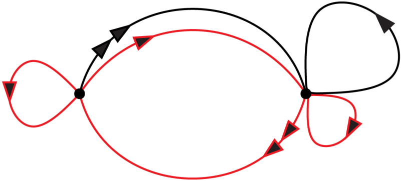

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2. Let $\{l_m\}$ be the increasing sequence of length of paths in G with initial vertex $i \in \mathcal{V}$. Then tiles of maximal volume in π_m are associated with paths of length l_m.

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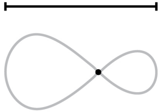

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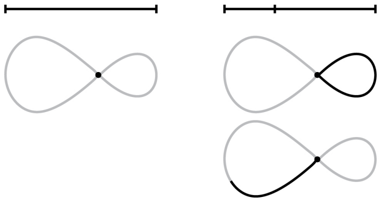

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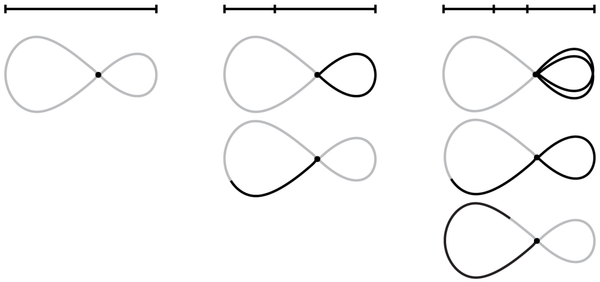

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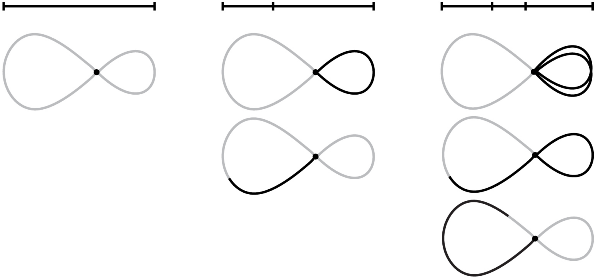

- Counting tiles in π_m is reduced to counting metric paths of length l_m in the associated graph.

Incommensurable and commensurable schemes

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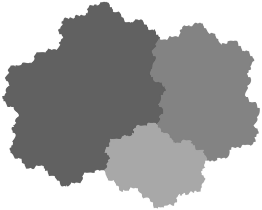
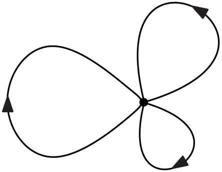
A commensurable example - **Rauzy fractal scheme:**





Edge lengths: $\log \tau, 2\log \tau, 3\log \tau$, where $\tau =$ tribonacci constant.

The incommensurable case - counting tiles

Theorem

Let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be a set of prototiles in $\mathbb{R}^d$ and let $\{\pi_m\}$ be a sequence of partitions of a tile $\mathcal{T}_i$ generated by an irreducible incommensurable multiscale substitution on $\mathcal{F}$. Then

$$|\{\text{Tiles} \in \pi_m\}| = \sum_{j=1}^{n} \sum_{h=1}^{n} \sum_{k=1}^{k_{hj}} \frac{1 - \left(\beta_{hj}^{(k)}\right)^d}{d} q_h e^{dl_m} + o\left(e^{dl_m}\right), \quad m \to \infty,$$

independent of i.

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1. Construction of graph associated with equivalent normalized scheme.
2. Metric path counting results from [Kiro, Smilansky×2 (2018)].
3. Special properties of graphs which are associated with substitution schemes.

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- For $m > m_0$

$$\frac{|\{x_m \cap T\}|}{|x_m|} = \frac{|\{\text{Tiles} \in \widetilde{\pi}_{m-m_0}\}|}{|\{\text{Tiles} \in \pi_m\}|} = \frac{e^{d\left(l_m - l_{m_0}\right)}}{e^{dl_m}} + o(1),$$

and since $e^{-l_{m_0} d} = \frac{\mathrm{vol}\, T}{\mathrm{vol}\, \mathcal{T}_i}$ uniform distribution follows.

The incommensurable case - types and their frequencies

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Under the previous assumptions

$$\frac{|\{\text{Tiles of type } r \text{ in } \pi_m\}|}{|\{\text{Tiles in } \pi_m\}|} = \frac{\sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} \left(1 - \left(\beta_{hr}^{(k)}\right)^d\right)}{\sum_{r=1}^{n} \sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} \left(1 - \left(\beta_{hr}^{(k)}\right)^d\right)} + o(1).$$

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The volume of the region covered by tiles of type r in π_m is

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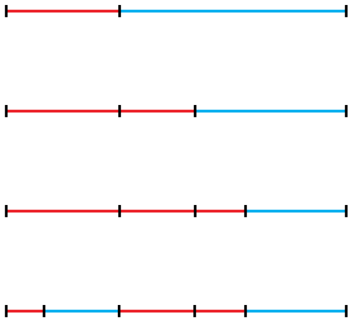
2. Special properties of graphs and the relevant probabilities which are associated with substitution schemes.

A nice answer

Back to the red-blue $\frac{1}{3}$-Kakutani sequence:



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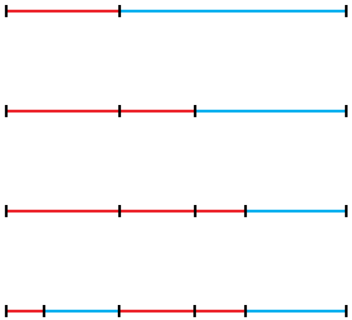

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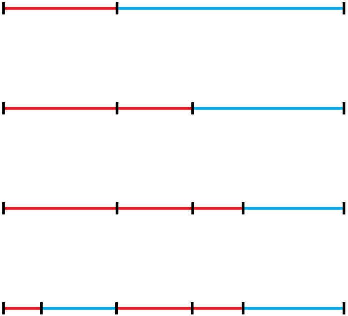

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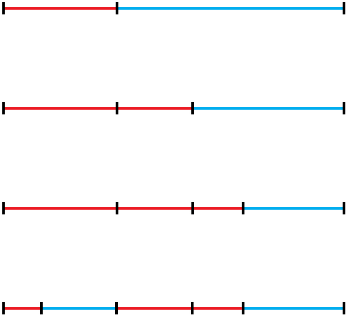

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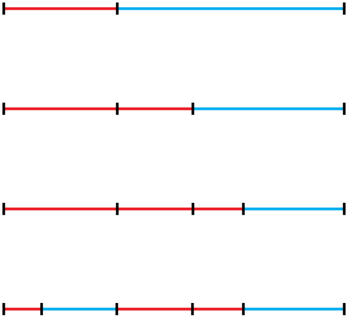

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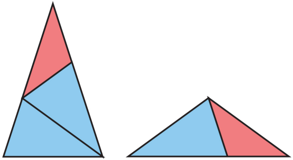

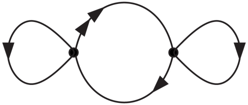

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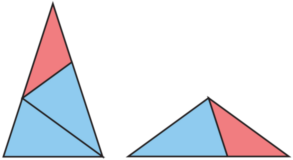
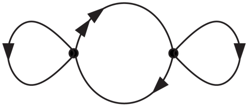
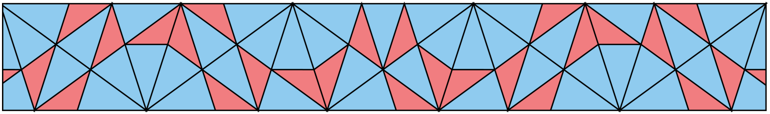




This is the classical setup for **substitution tilings** of $\mathbb{R}^d$:



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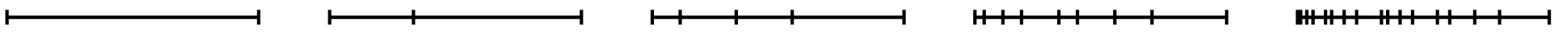

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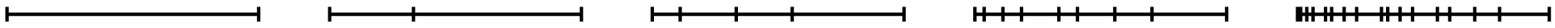

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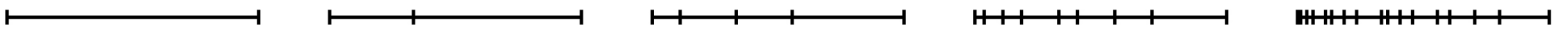

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Follows from the Perron-Frobenius Theorem for irreducible matrices, and additional standard results on cyclic matrices.

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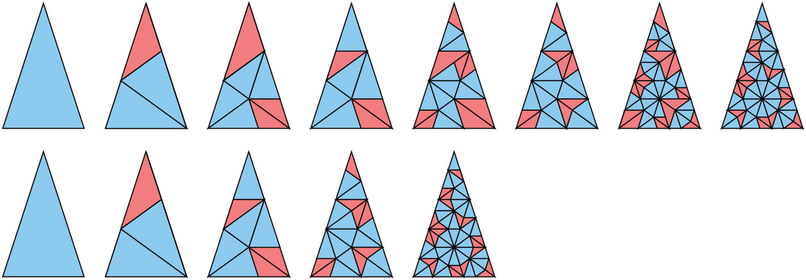

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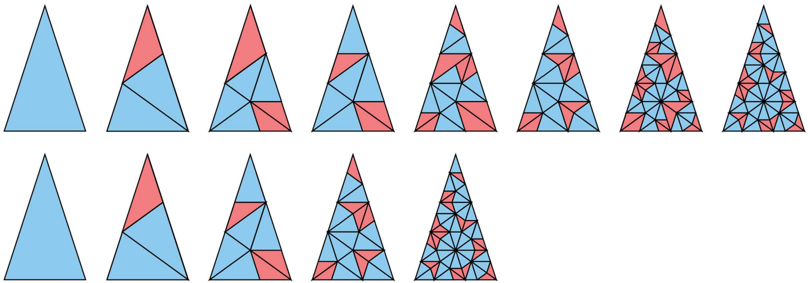

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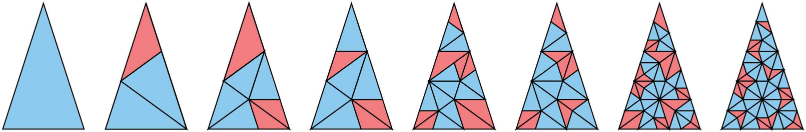

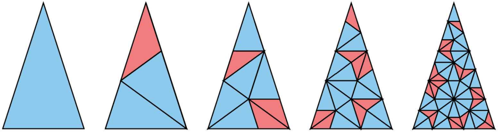

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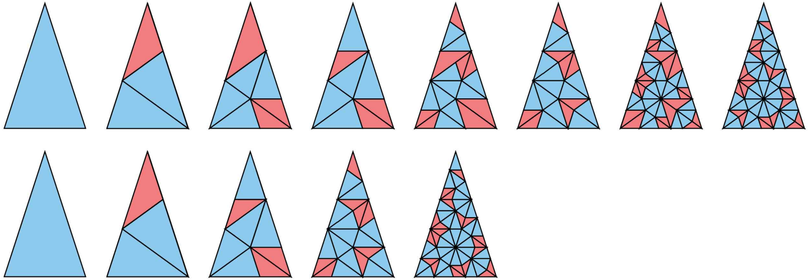

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The lemma is proved by applying a “slowing down” process.

Multiscale substitution tilings (with Yaar Solomon)

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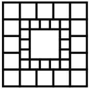
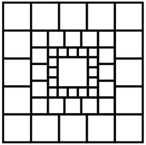
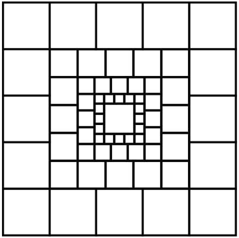
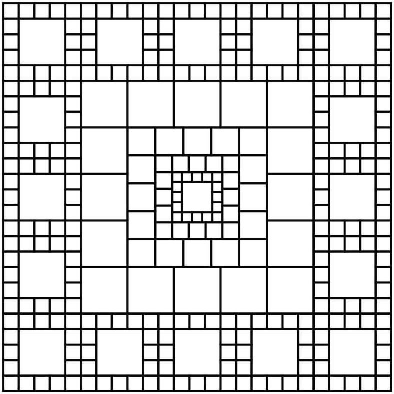
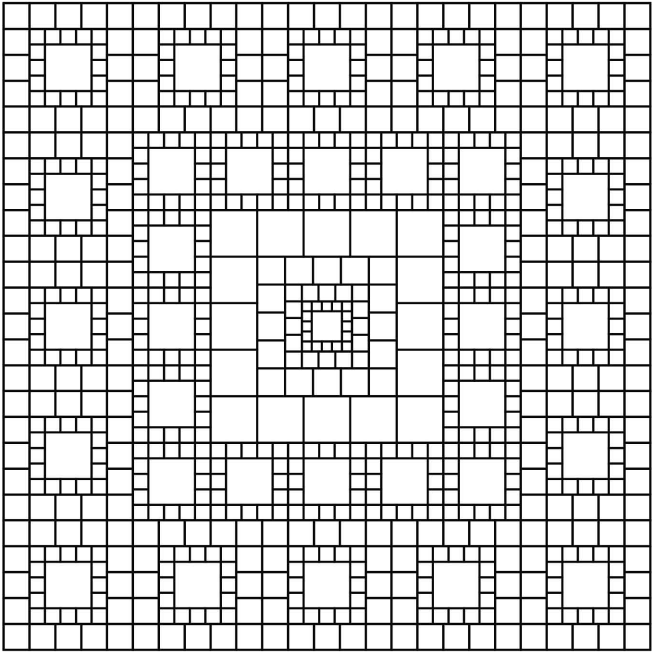
Elements of X_H are called **multiscale substitution tilings**.

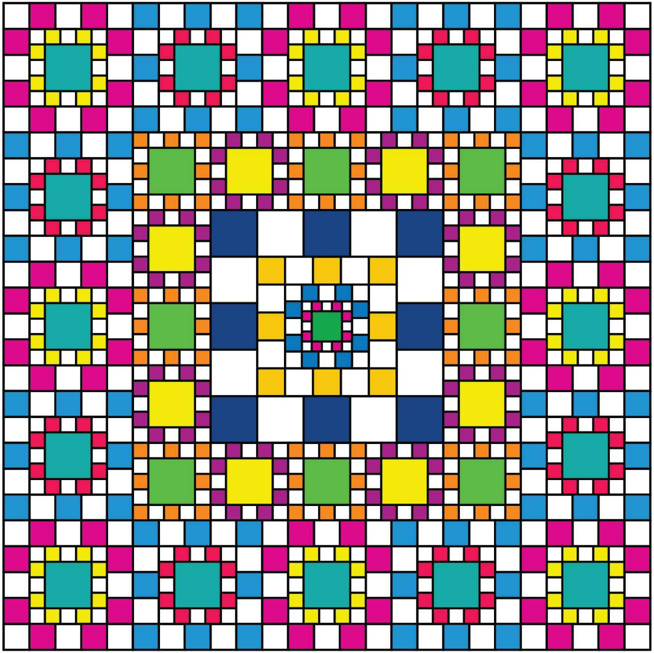












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- Many more beautiful properties! *Coming soon...*

Thanks!